Decisions with partially and imprecisely specified judgements

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Utility functions and prior beliefs — Experts

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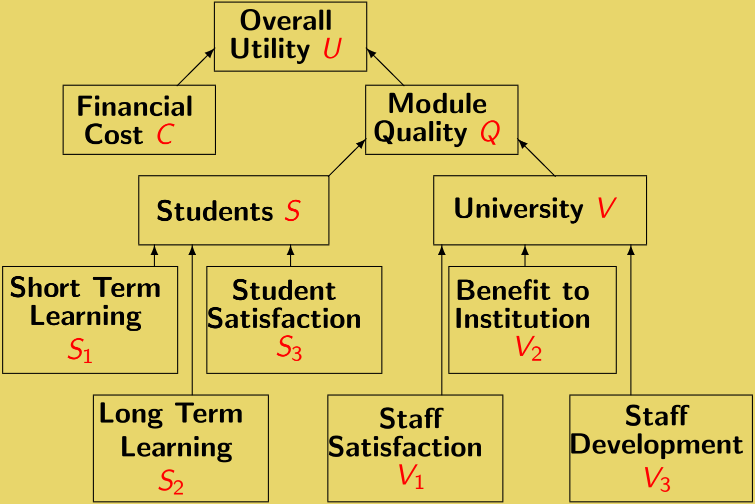
- Partial belief specification.

Expert opinion in decision making

1. Suitable structures for multi-attribute utility functions.
2. Requisite expectations for evaluation of overall expected utility.
3. Elicitation.
4. Imprecise specifications.
5. Choosing decisions, sensitivity.

Utility hierarchy: Course design



Imprecise utility: Introduction

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    - Farrow and Goldstein (2006) *etc.*
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  - See Farrow (2013).

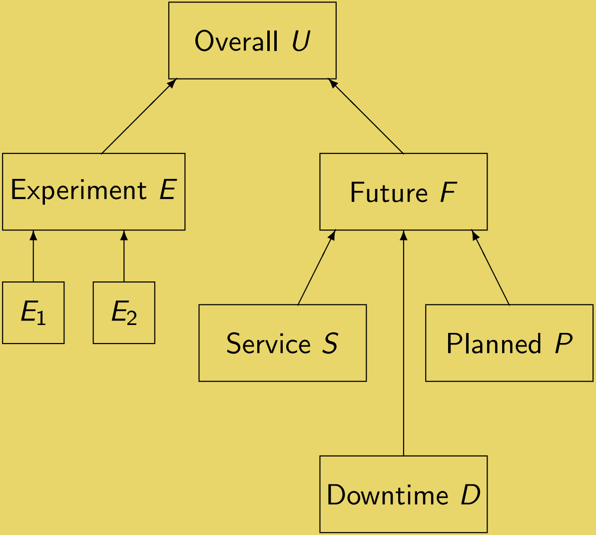
Imprecise Utility

- Imprecise trade-offs.
- Imprecise marginal utility functions.

Imprecise Utility

- Imprecise trade-offs.
- Imprecise marginal utility functions.
- Possible extension: imprecise expectations.
  - Lower and upper previsions
  - Walley (1991)
  - Troffaes and de Cooman (2014).

Life-testing experiment utility hierarchy



Structure: Utility Hierarchy

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Structure: Utility Hierarchy

- Utility hierarchy
- At each node we have mutual utility independence over parents.
  - This allows a finite parameterisation of the combined utility function.
- All utilities are on a standard scale.
  - Worst outcome considered: $U = 0$.
  - Best outcome considered: $U = 1$.

  This allows us to interpret utilities and trade-offs at all nodes.

Combining utilities at child nodes

- Additive node

$$U = \sum_{i=1}^{s} a_i U_i$$

with $\sum_{i=1}^{s} a_i \equiv 1$ and $a_i > 0$ for $i = 1, \ldots, s$.

- Binary node

$$U = a_1 U_1 + a_2 U_2 + h U_1 U_2$$

where $0 < a_i < 1$ and $-a_i \leq h \leq 1 - a_i$, for $i = 1, 2$, and $a_1 + a_2 + h \equiv 1$.

Combining utilities at child nodes

- Multiplicative node

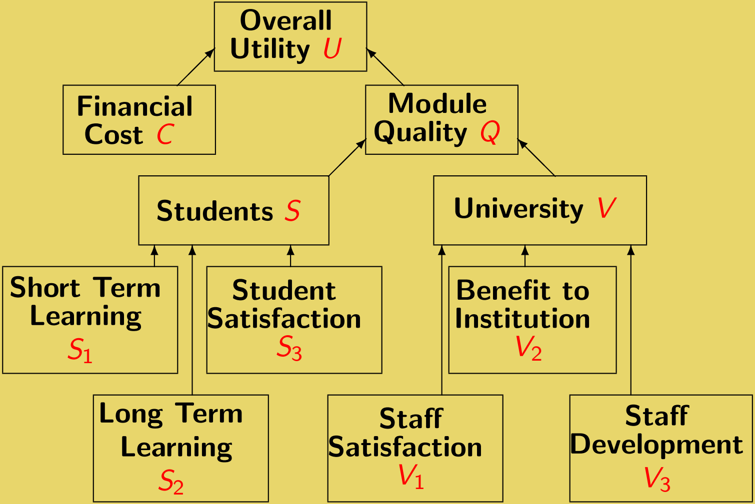
$$U = B^{-1}\left\{\prod_{i=1}^{s}[1 + ka_i U_i] - 1\right\}$$

with

$$B = \prod_{i=1}^{s}(1 + ka_i) - 1$$

$a_1 \equiv 1, k > -1$ and, for $i = 1, \ldots, s,$ we have
$a_i > 0, \quad ka_i > -1.$

Utility hierarchy: Course design



Imprecise Utility Tradeoffs

Example: Course Design, Node Q: Module Quality.

$$U_Q = a_S U_S + a_V U_V + h_Q U_S U_V$$

Imprecise Utility Tradeoffs

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$$U_Q = a_S U_S + a_V U_V + h_Q U_S U_V$$

Choose

	Attribute values such that:
Either (A)	$U_S = 1,\ U_V = 0$ with certainty
Or (B)	$U_S = U_V = 1$ with probability α
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(A) preferred when $\alpha < 0.50$ so $a_S \geq 0.5$.
(B) preferred when $\alpha > 0.89$ so $a_S \leq 0.89$

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Either (A)	$U_S = 1,\ U_V = 0$ with probability $1/3$
	$U_S = 0,\ U_V = 1$ with probability $2/3$
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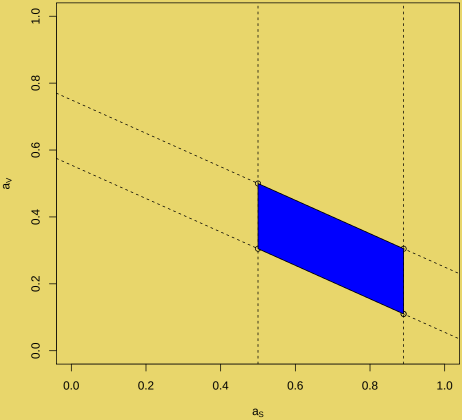
	$U_S = U_V = 0$ with probability $1 - \alpha$

(A) preferred when $\alpha < 0.37$ so $a_V \geq 0.555 - a_S/2$.
(B) preferred when $\alpha > 0.50$ so $a_V \leq 0.75 - a_S/2$

Elicitation and feasible set: Binary node



Analysis

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  - Reduce the number of choices to be considered.
  - Select a proposed choice d^*.
  - Identify the nodes and trade-offs responsible for the elimination of choices.
- Examine sensitivity
  - Farrow and Goldstein (2010).
  - Boundary linear utility
  - Volumes and distances

Imprecision in risk aversion

- Z a scalar attribute scaled so that $0 \leq Z \leq 1$.
- Direct method:
  - Determine a range for $U(z^*)$ where $0 < z^* < 1$.
  - Probability equivalent method.
  - Offer the decision maker a choice between
    - d_A : the attribute value corresponding to $z = z^*$, with certainty, and
    - d_B : with probability α, the attribute value corresponding to $z = 1$ and, with probability $1 - \alpha$, the attribute value corresponding to $z = 0$.
  - The lower utility for z^*, $U_1(z^*)$ is the largest value of α at which the decision maker would choose d_A.
  - The upper utility for z^*, $U_2(z^*)$ is the smallest value of α at which the decision maker would choose d_B.

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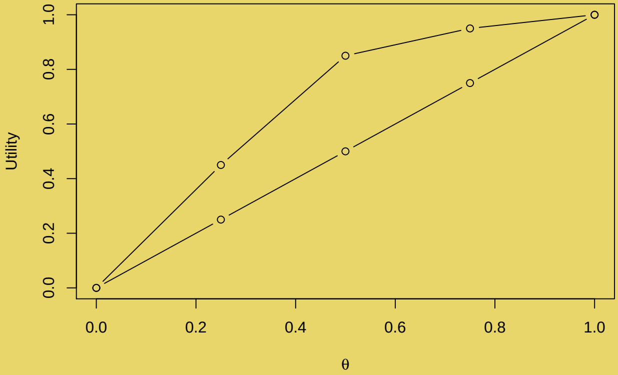
  - The upper utility for z^*, $U_2(z^*)$ is the smallest value of α at which the decision maker would choose d_B.
- Repeat this process at a range of values z^*.
- Interpolate (linear?). Obtain lower and upper utility functions, $U_1(z)$ and $U_2(z)$.
- These can then be our two basis functions.

Example — Imprecise marginal utility



Imprecision in risk aversion

- Possibility of additional basis functions to give more flexibility in shape.
- *Eg* one which is closer to $U_1(z)$ for some of the range of z and otherwise closer to $U_2(z)$.

Partial belief specification: Bayes linear methods

- Book: Goldstein and Woof (2007)
- Collection of unknowns. Split into two subvectors X, Y.
- Specify means, variances, covariances:

$$\mathrm{E}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} m_x \\ m_y \end{pmatrix}, \quad \mathrm{Var}\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} V_{xx} & V_{xy} \\ V_{yx} & V_{yy} \end{pmatrix}$$

Bayes linear methods

Bayes linear methods

If we observe X:
adjusted mean and variance of Y:

$$
\begin{aligned}
\mathrm{E}_{Y|X}(Y \mid X = x) &= m_y + V_{yx} V_{xx}^{-1} (x - m_x), \\
\mathrm{Var}_{Y|X}(Y \mid X = x) &= V_{yy} - V_{yx} V_{xx}^{-1} V_{xy}.
\end{aligned}
$$

- Alternative representation

$$
\begin{aligned}
\mathrm{E}(X) &= m_X, \qquad \mathrm{Var}(X) = V_{XX}, \\
Y &= m_y + M_{Y|X}(X - m_x) + U_{Y|X}, \\
\mathrm{E}(U_{Y|X}) &= \underline{0}, \qquad \mathrm{Var}(U_{Y|X}) = V_{Y|X}.
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- So

$$
\begin{aligned}
\mathrm{E}(Y) &= m_Y, \\
\mathrm{Var}(Y) &= M_{Y|X} V_{XX} M_{Y|X}^T + V_{Y|X}, \\
\mathrm{Covar}(Y, X) &= M_{Y|X} V_{XX}.
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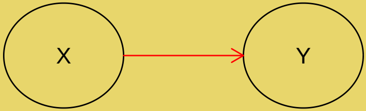
- Same as before if

$$\begin{aligned} M_{Y|X} &= V_{YX} V_{XX}^{-1}, \\ V_{Y|X} &= \mathrm{Var}(Y \mid X = x) = V_{YY} - V_{YX} V_{XX}^{-1} V_{XY}. \end{aligned}$$

Bayes linear methods



Example: Elicitation — lifetime distribution

$$T \mid \lambda \sim \mathrm{Exp}(\lambda)$$

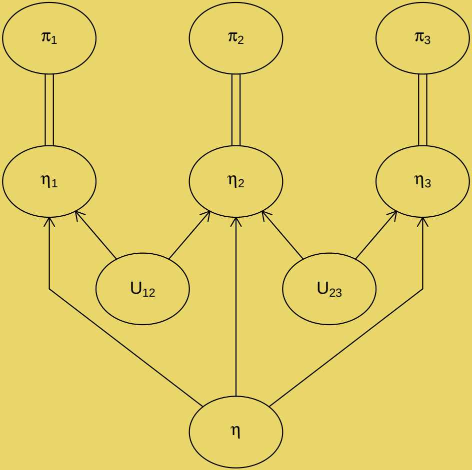
What proportion, π would fail before time τ?

$$\pi = 1 - \exp(-\lambda\tau)$$

$$\eta = \log\lambda = \log\left[-\frac{\log(1-\pi)}{\tau}\right]$$

Three experts give point assessments of π.

Example: Three Experts



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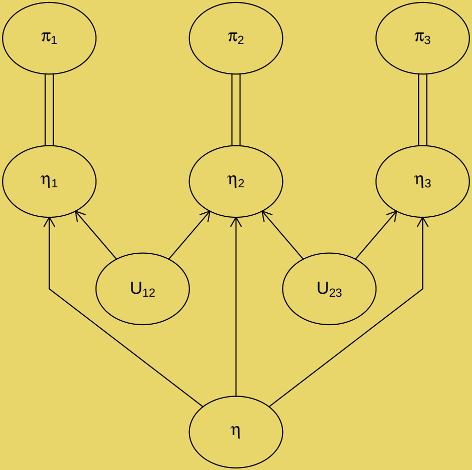

Common and specific uncertainty factors: Farrow (2003).

- More generally, what if we don't get point values which we treat as observations from experts but information which causes us to change our mean and variance for η?
- For example, we elicit an interval for η.

Bayes linear kinematics

$$Y = m_y + M_{Y|X}(X - m_x) + U_{Y|X} \qquad (1)$$

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- If so: Bayes linear kinematics, Goldstein and Shaw (2004) (*cf* probability kinematics: Jeffrey, 1965).

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- See also
  - Wilson and Farrow (2010) – failure times
  - Gosling *et al.* (2013)
  - Wilson and Farrow (2017) – survival model
  - Wilson and Farrow (*in prep*) – design

- Are successive belief updates for $B = X \cup Y$ by $D_1, D_2, \ldots$ commutative?
- Goldstein and Shaw (2004): under certain conditions the commutativity requirement leads to a unique BLK update:

$$V_1^{-1}(B) = \mathrm{Var}^{-1}_{B|D_1,\ldots,D_s}(B \mid D_1, \ldots, D_s) = V_B^{-1}(B) + \sum_{k=1}^{s} P_k(B)$$

where

$$P_k(B) = \mathrm{Var}^{-1}_{B|D_k}(B \mid D_k) - V_B^{-1}(B)$$

and

$$V_1^{-1}(B)\mathrm{E}_{B|D_1,\ldots,D_s}(B \mid D_1, \ldots, D_s) = V_B^{-1}(B)\mathrm{E}(B) + \sum_{k=1}^{s} F_k(B)$$

where

$$F_k(B) = \mathrm{Var}^{-1}_{B|D_k}(B \mid D_k)\mathrm{E}_{B|D_k}(B \mid D_k) - V_B^{-1}(B)\mathrm{E}(B)$$

Bayes linear Bayes graphical model

- Goldstein and Shaw (2004)
- Bayes linear belief structure for $B = \{Y, X_1, \ldots, X_s\}$ where $Y, X_1, \ldots, X_s$ are (vector) unknowns.
- Full (Bayesian) probability specification for each of $(X_1, D_1), \ldots, (X_s, D_s)$.
- Given X_j , D_j is conditionally independent of everything in $\{Y, X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_s, D_1, \ldots, D_{j-1}, D_{j+1}, \ldots, D_s\}$.
- Use of transformation — Wilson and Farrow (2010).
- Non-conjugate updates — Wilson and Farrow (*in future*).

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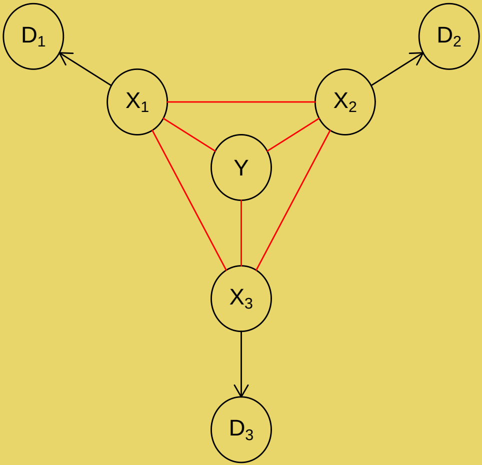

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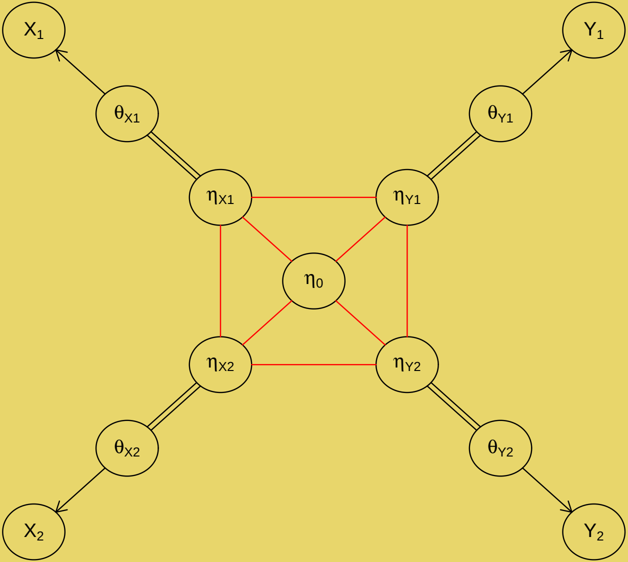

Application to Expert Judgement

- Example as before: $\pi = \mathsf{Pr}(T < \tau)$.

$$\eta = \log\left[-\frac{\log(1-\pi)}{\tau}\right]$$

- Now suppose each expert specifies quartiles.

Application to Expert Judgement: Possible method

- Fit $\mathrm{Beta}(a_i, b_i)$ distribution to quartiles of Expert i.

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$$L_i \propto \pi^{a_i - 1}(1 - \pi)^{b_i - 1}$$

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- Combine with $\text{Beta}(a_{0,i}, b_{0,i})$ prior for Expert i's judgement about π.
  Posterior: $\text{Beta}(a_{1,i}, b_{1,i})$ where
  $a_{1,i} = a_{0,i} + a_i - 1, \; b_{1,i} = b_{0,i} + b_i - 1$.

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  $a_{1,i} = a_{0,i} + a_i - 1,\ b_{1,i} = b_{0,i} + b_i - 1$.
- Propagate through Bayes linear Bayes structure using Bayes linear kinematics.

Application to Expert Judgement: Possible method

- This is work in progress!
- Should an expert who gives a more precise interval have so much more effect?
- Possible refinement:

  Let

  $$p_i = \frac{a_i}{a_i + b_i}$$

  Use likelihood

  $$\tilde{L}_i \propto \pi^{m_i p_i - 1}(1-\pi)^{m_i(1-p_i)-1}$$

  where

  $$m_i = g(n_i) < n_i = a_i + b_i.$$

Summary

- Structure for multi-attribute utility.
- Imprecision in trade-offs.
- Imprecision in marginal utilities.
- Identify required expectations.
- Include imprecision in expectations (future)?
- Moment-based belief elicitation using Bayes linear kinematics and Bayes linear Bayes models — probability distributions not fully specified.

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